On Semi-C-Reducible Finsler Metrics

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Abstract--- *The class of semi-C-reducible manifolds contains the class of Randers manifolds and Landsberg manifolds as special cases. In the present paper, we showed that every semi-C-reducible manifold with C-reducible metric reduces to a Landsberg manifold. And also we proved that there is no existence of $C2$-like Randers metric.*

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I. INTRODUCTION

In Finsler geometry, there are several non-Riemannian quantities, the Cartan torsion C, the Berwald curvature B, the Landsberg curvature L, the mean Landsberg curvature J and the stretch curvature $\sum$ etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. The study of these quantities is benifit for us to make out their distinction and the nature of Finsler geometry.

The study of the reducibility of the Cartan tensor C_{hij} in Finsler spaces was initiated by Matsumoto, in order to explain different curvature and torsion tensors explicitly. Further C-reducible, semi-C-reducible, quasi-C-reducible and C^{v}-reducible Finsler spaces have been studied by various authors [4] [5] [6] [8] [10].

Let $[M, L(x,y)]$ be an n-dimensional Finsler space $(n \geq 3)$, where M is an n-dimensional differentiable manifold endowed with the fundamental function $L = L(x,y)$, let $x = (x^i)$ be a point of M and let $y = (y^i)$ be a supporting element of M. The metric tensor g_{ij} of L is given by,

$$g_{ij}^{'} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$$

A tensor field of (r,s)-type on TM is called Finsler tensor field (or a d- tensor field) if under a change of the induced co-ordinates on TM its components transform like the components of a (r,s)-type tensor on the base manifold.

It is very easy to see that gij are the components a $(0,2)$-type Finsler tensor field and is called the fundamental tensor or the metric of the Finsler space. From this we have another important Finsler tensor field,

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k},$$

Which is called the Cartan tensor. Two other important tensors in Finsler geometry are $C_i = C_{ijk}\, g^{jk}$ the so-called torsion vector. And $h_{ij} = L\, \frac{\partial^2 L^2}{\partial y^i \partial y^j}$, the angular metric tensor. Moreover the angular metric tensor h_{ij} can be written in terms of the normalized supporting element $l_i = \frac{g_{ij} y^j}{L}$, as $h_{ij} = g_{ij} - l_i l_j$.

Matsumoto and H$\bar{o}$j$\bar{o}$ worked a conclusive theoremn on C-reducible Finsler spaces [7].

Narasimhamurthy S. K. et al studied on v-curvature tensor of $C3$-like conformal Finslerspaces [10]. A. Tayebi et al worked on semi-P-reducible Finsler metrics and also $C3$-like Finsler metrics [11] [12]. And also there are so many authors were studied on Reducible special Finsler spaces.

Let L be n-dimensional Finsler space equipped with metric function $L(x,y)$. In the geometry of Finsler spaces based on of a tensor field, we shall denote $|i$, as the v-covariant differentiation and $\dot{\partial}_j$ as the δ- differentiation.

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That is, the partial differentiation with respect to the element support there are three curvature tensors and three torsion tensors of Cartan's connection $c\Gamma$. These are :

1) R_{hijk} h- curvature tensor.
2) P_{hijk} hv- curvature tensor.
3) S_{hijk} v- curvature tensor.
4) $R_{ijk} = Y^h R_{hijk}$ $(v)h$- torsion tensor.
5) $P_{ijk} = Y^h P_{hijk}$ $(v)hv$- torsion tensor.
6) $C_{ijk} = \frac{1}{4}\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2$ $(h)hv$- torsion tensor.

Various interesting forms of these curvature tensors and torsion tensors have been studied by Matsumoto and others. Two of them are C-reducible Finsler space and P-reducible Finsler space, in which the $(h)hv$-torsion tensor and $v(hv)$ −torsion tensor are of the form,

$C_{ijk} = \frac{1}{(n+1)}(c_i h_{jk} + c_j h_{ki} + c_k h_{ij})$, $P_{ijk} = \frac{1}{(n+1)}(P_i h_{jk} + P_j h_{ki} + P_k h_{ij})$ respectively, here h_{ij} is the angilar metric tensor, $C_i = C_{ijk}\, g^{jk}$ and $P\_i = P_{ijk}\, g^{jk}$. It is to be noted that fundamental functionb of any C-reducible Finsler space is of the Randers type or the Kropina type. Every C-reducible Finsler space is P-reducible and converse is not necessarily true.

The purpose of the present paper is to consider a special form of C_{ijk} that is, Semi-C-reducible Finsler space. We have study the L reduces to a Landsberg metric and also we study there does not exists any $C2$-like Randers metric.

II. PRELIMINARIES

Definition 2.1: A Finsler metric is a scalar field $L(x,y)$ which satisfies the following three conditions:

i. It is defined and differential for any point of $TM \setminus \{0\}$,
ii. It is positively homogeneous of first degree in y^i, that is,
$L(x,\ \lambda y) = \lambda L(x,\ y)$, for any positive number λ,
iii. It is regular, that is,
$g_{ij}(x,\ y) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$,

Constitute the regular matrix g_{ij}, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The manifold M equipped with a fundamental function $L(x,y)$ is called Finsler space $F^n = (M,L)$.

We call the indicatrix in $x\epsilon M$, the hypersurface S_x in $T_x M$ defined by the equation $L(x,y) = 1$ and denote by SM the fiber bundle of unitary tangent vectors to M. We obtain a symmetric tensor C, Cartan tensor, on $\pi^* TM$ defined by,

$$C(u,v,w) = C_{ijk}\, u_i u_j w_k,$$

where $u = u_i \frac{\partial}{\partial x^i}$, $v = v_j \frac{\partial}{\partial x^j}$, $w = w_k \frac{\partial}{\partial x^k}$ and $\frac{C_{ijk}\ 1}{4}\ L^2]_{y_i y_j y_k}$. The family $C = C_{y\epsilon T M_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if L is Riemannian.

For $y\epsilon TM_0$, define mean Cartan torsion I_y by $I_y(u) = I_i(y)u^i$, where

$$I_i = g^{jk}\, C_{ijk},$$

g^{jk} is the inverse of g_{jk} and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Deicke Theorem, L is Riemannian if and only

if $I_y = 0$.

Given a Finsler manifold (M,L), then a global vector field G is induced by L on TM_0, which in a standard co-ordinate x^i, y^i for TM_0 is given by $G = y^i \frac{\partial}{\partial x^i} - 2G(x,y)\ \frac{\partial}{\partial y^i}$, where

$$G^i(x,y) = \frac{1}{4} g^{il}(x,y)\ \ \frac{\partial^2 L^2}{\partial x^k \partial y^l} y^k - \frac{\partial L^2}{\partial x^l}\Big\},$$

are called the spray coefficient of G. then G is called the spray associated to (M,L). In local co-ordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $c^i(t)$ satisfy,

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right. = 0.$$

Let (M,L) be a Finsler manifold. For $y\epsilon T_x M_0$, define the Matsumoto torsion

$$M_y : T_x M \otimes T_x M \to \mathbb{R} \text{ by } M_y(u,v,w) = M_{ijk}\, u^i v^j w^k$$

where

$$M_{ijk} = C_{ijk} - \frac{1}{n+1} I_i h_{jk} + I_j h_{ki} + I_k h_{ij},$$

$h_{ij} = L_{y_i y_j} = g_{ij} - \frac{1}{L^2} g_{ip} y^p g_{jq} y^q$ is the angular metric. A Finsler metric L is said to be C-reducible metric if $M_y = 0$.

Definition 2.2: A non-Riemannian Finsler space L of dimension $n \geq 3$ is called C-reducible if the $(h)hv$-torsion tensor C_{ijk} is written is the form,

$$C_{ijk} = \frac{1}{n+1}\{c_i h_{jk} + c_j h_{ki} + c_k h_{ij}\},$$

where $c_i = c^j_{ij}$.

Definition 2.3: A Finsler space L, $n_- \geq 2$ with $c^2 = g^{ij} c_i c_j \neq 0$ is called $C2$-like, if the $(h)hv$-torsion tensor C_{ijk} is written in the form,

$$C_{ijk} = \frac{1}{c^2} c_i c_j c_k.$$

Definition 2.4: A Finsler space L, $(n \geq 3)$ with the non-zero length C of the torsion vector C^i is called semi-C-reducible, if the $(h)hv$-torsion tensor C_{ijk} is of the form,

$$C_{ijk} = \frac{p}{n+1}\{c_i h_{jk} + c_j h_{ki} + c_k h_{ij}\} + \frac{q}{c^2} c_i c_j c_k,$$

where p and $q = 1 - p$ do not vanish. p is called the characteristic scalar of the L.

III. MAIN RESULTS

In this present section, we considered the semi-C-reducible Finsler manifold with C-reducible metric. Then showed the following results:

Theorem 3.1: Let (M, L) be a semi-C-reducible Finsler manifold where p and $q = (1 - p)$ donot vanish. Suppose that L is a C-reducible metric. Then L reduces to a Landsberg metric.

Proof: Let L is a C-reducible metric,

$$C_{ijk} = \frac{1}{n+1}\{c_i h_{jk} + c_j h_{ki} + c_k h_{ij}\} \quad (3.1)$$

On the other hand, L is a semi-C-reducible metric,

$$C_{ijk} = \frac{p}{n+1} \quad c_i h_{jk} + c_j h_{ki} + c_k h_{ij}\} + \frac{q}{c^2} c_i c_j c_k, \quad (3.2)$$

where, p and $q = (1 - p)$ donot vanish, p is called the characteristic scalar of the L. From Eq. (3.1) and (3.2) we have

$$\frac{1}{n+1} \quad 1 - p)\{c_i h_{jk} + c_j h_{ki} + c_k h_{ij}\} = \frac{q}{c^2} c_i c_j c_k. \quad (3.3)$$

Eq. (3.3) is multiplied by g^{ij} which implies,

$$\frac{1}{n+1}(1 - p) - q\Big\} c_k = 0. \quad (3.4)$$

Assume that $c_k \neq 0$. So we get,

$$p = 1 - q(n+1). \quad (3.5)$$

Substituting Eq. (3.5) into Eq. (3.2) we have the following,

$$C_{ijk} = \frac{1}{n+1}\{c_i h_{jk} + c_j h_{ki} + c_k h_{ij}\} - q\left\{[c_i h_{jk} + c_j h_{ki} + c_k h_{ij}] - \frac{1}{c^2} c_i c_j c_k\right\}. \quad (3.6)$$

Since, L is a C-reducible metric, thus Eq. (3.6) reduces to the following,

$$q\left\{[c_i h_{jk} + c_j h_{ki} + c_k h_{ij}] - \frac{1}{c^2} c_i c_j c_k\right\} = 0. \quad (3.7)$$

From Eq. (3.7) and our assumptions, we derive that the following holds,

$$c_i h_{jk} + c_j h_{ki} + c_k h_{ij} = \frac{1}{c^2} c_i c_j c_k. \quad (3.8)$$

Which is not possible, since $Rank(h_{jk} c^2) = n - 1$ and $Rank(c_j c_k) = 1$. Thus,

$$c^2 = c_i c^i = 0. \quad (3.9)$$

Since, L is a positive-definite metric, then $c_i = 0$. Thus by Eq. (3.1), we conclude that L is a Landsberg metric.

Theorem 3.2: Suppose (M, L) be a Finsler manifold of dimension $n \geq 3$. Then there does not exists any $C2$-like Randers metric.

Proof: Consider L be a $C2$-like Randers metric on a manifold M of dimension $n \geq 3$. It is easy to see that L is C-reducible. We have the following,

$$c_i h_{jk} + c_j h_{ki} + c_k h_{ij} = \frac{1}{c^2} c_i c_j c_k. \quad (3.10)$$

We have,

$$h_{ij} c^i = (g_{ij} - l_i l_j) c^i = c_j. \qquad (3.11)$$

Contract Eq. (3.10) with ci and using Eq. (3.11) we get,

$$c^2 h_{jk} + 2c_j c_k = c_j c_k, \qquad (3.12)$$

or equivalently we can write

$$c^2 h_{jk} = -c_j c_k. \qquad (3.13)$$

By the same argument used in Theorem 3.1, we conclude that there does not exists any $C2$-like Randers metric on a manifold M of dimension $n \geq 3$. Hence proof is completed.

IV. CONCLUSION

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians. The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg. Matsumoto introduces the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible.

In this research article, we worked on special Finsler space semi-C-reducible. Here we studied, if L is semi-C-reducible Finsler metric then L reduces to a Landsberg metric. Further worked there is no existence of any $C2$-like Randers metric.

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